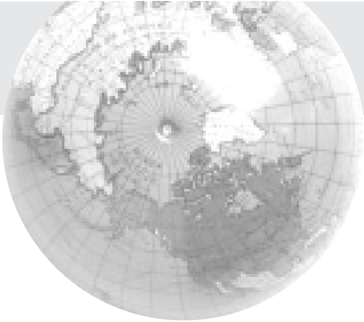
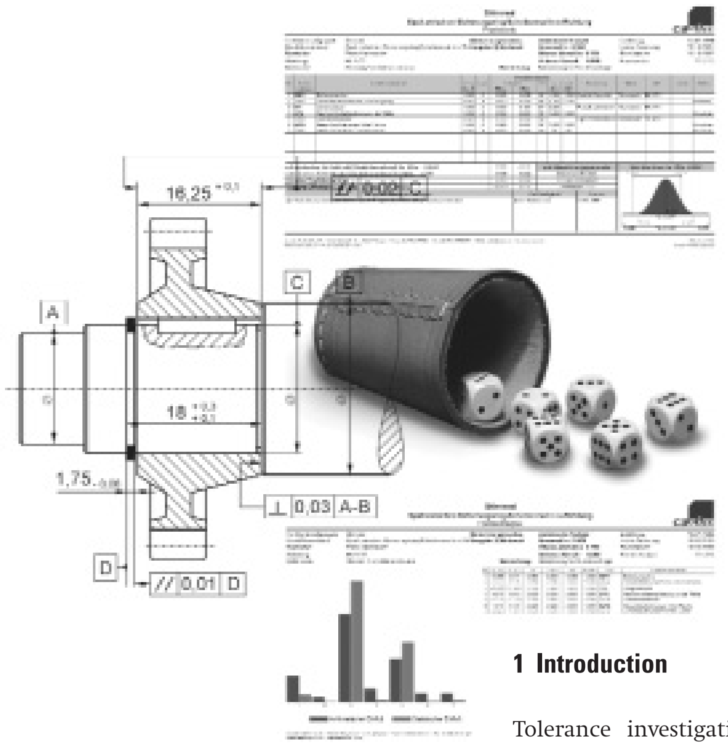


You will find the figures mentioned in this article in the German issue of ATZ 10/2005 beginning on page 896.

Toleranzanalysen an beliebigen physikalischen Systemen

Tolerance Analyses of Arbitrary Physical Systems



Tolerance analyses are indispensable to prove the technical practicability of sub-assemblies under mass production conditions. This contribution is intended as an attempt to show an empirical resolution method which allows the easy implementation of complex tolerance analyses on arbitrary physical systems.

Author:
Frank Mannewitz

1 Introduction

Tolerance investigations, frequently also termed tolerance analyses, have been indispensable ever since the beginning of the last century, starting with Henry Ford's pioneering work. Their purpose is to prove the technical practicability of sub-assemblies under mass production conditions. However, in recent years the subject of tolerance analyses on technical sub-assemblies has gained more and more significance, not least due to the fact that tolerance analyses can determine tolerance margins and, by implication, manufacturing costs.

The calculation of manufacturing tolerances has become a central theme for the motor vehicle industry and its sub-contractors alike, as the competitiveness of a company is decided by quality requirements as well as time requirements

For a tolerance analysis to be carried out, the functional interdependencies of a sub-assembly or technical system must be known. That is to say that it must also be known which links form the closed chain and, above all, what influence each individual link of the chain has on the function of the sub-assembly.

Besides describing the general methods used in tolerance analysis, this contribution is intended to demonstrate an empirical resolution method for determining linearity coefficients. This method consists of simple experiments to show functional interdependencies quickly and easily and thus facilitate the implementation of tolerance analyses. This paper will, therefore, include statistical analysis as well as arithmetical analysis.

2 Description of Closed Dimension Chains

Figure 1 shows part of the forward section of a metal housing used for adjusting an automobile air conditioning system. The illustration depicts the bearing of the drive- and transmission cog for the ventilation flap. In general, transmission of force in a gear system (in this case an involute gear) can only be guaranteed if the meshing of the two wheels is harmonised in such a way that the sum of the two pitch circle radii is equal to the dimension required M_0 between the centre of the shafts.

Therefore the dimension M_0, also called closure dimension, necessary to ensure functionality is fixed at 47.4 ± 0.3 mm. If the closure dimension is too small, the teeth heads can protrude into the teeth bases of the opposite cog. If the closure dimension is too large, the cogs may not run smoothly or even fail to interlock entirely.

The tolerances for form and position normally required for a correct function description are not taken into account in the calculation example. This should be taken into account, based on the appropriate tolerance principle and tolerance type.

The individual dimensions which affect the distance between shafts M_0 in this case are the horizontal and vertical positions of the bearing centres in the metal housing structure relative to their reference faces.

Since the technical implementation of the example shown here should be realised with a double perforation operation, it is necessary to consider the respective X- and Y-coordinates separately. These are a result of the tool precision on the one hand, and of the exact insertion of the metal into the tool on the other. The holes for the drive wheel bearing are punched during the first stage and those fort he transmission wheel during the second. Consequently the effective tolerated individual dimensions of the distance between shafts M_0 are the horizontal lengths M_1 and M_2 and the vertical lengths M_3 und M_4.

So the nominal dimension of the distance between the two shafts under the nominal dimensions given in Figure 1 is calculated according to Pythagoras' theorem following Eq. (1) to N_0 = 47.413 mm. Eq. (1) describes (allowing for the respective tolerated individual dimension Mi) the objective function of the sought dimension M_0, as Eq. (2) shows. Generally speaking a closed dimension chain is a function of the respective tolerated individual dimensions, $M_0 = f(M_1, M_2,..., M_k)$.

$$N_0 = \sqrt{(N_2 - N_1)^2 + (N_4 - N_3)^2} \quad \text{Eq. (1)}$$

$$M_0 = \sqrt{(M_2 - M_1)^2 + (M_4 - M_3)^2} \quad \text{Eq. (2)}$$

If the closed dimension chain is a flat or spatially closed vector, it is usually referred to as multidimensional or non-linear dimension chain. In this case the non-linear influences of the respective tolerated individual dimensions M_i on the closure dimension M_0 are recorded via so-called linearity coefficients α_i. If the linearity coefficients of a dimension chain are exclusively ± 1, it is known as a one-dimensional or linear dimension chain [1]. So a dimension chain in general can be described by Eq. (3).

$$M_0 = \sum_{i=1}^{k} \alpha_i \cdot M_i \quad \text{Eq. (3)}$$

The general resolution method to determine linear coefficients is given in the linearisation of functions using the "total differential"[6]. In this case the function $y = f(x_1, x_2,..., x_k)$ in the immediate vicinity of the centre of area $P(x_0, y_0, z_0)$is replaced by a linear function, that is to say, the total differential of the function, see Eq. (4).

$$\Delta y = \sum_{i=1}^{k} \left(\left. \frac{\partial f}{\partial x_i} \right|_{x_{0_i}} \cdot \Delta x_i \right) \quad \text{Eq. (4)}$$

$$dM_0 = \sum_{i=1}^{k} \left(\left. \frac{\partial M_0}{\partial x_i} \right|_{N_1,...,N_k} \cdot dx_i \right) \quad \text{Eq. (5)}$$

The first order partial deductions are formed for the centre of area P. Also the Δx_i are the "minor" deviations with reference to the centre of area. As a result, Eq. (5) follows Eq. (4) for the linearisation of the objective dimension $M_0 = f(M_1, M_2,..., M_k)$. Subsequently the change of M0 results from the product of the respective linearity coefficient and the individual tolerance. Therefore the linearity coefficients can be calculated using Eq. (6). The respective nominal dimensions are considered in the derivations. Using Eq. (6), the following deductions result for the adjustment mechanism example according to equations (7) to (10), and therefore linearity coefficients α_1 = -0.46400699, α_2 = 0.46400699, α_3 = -0.88583153 and α_4 = 0.88583153. When the linearity coefficients are known, the nominal closure dimension N_0 may be calculated according to Eq. (11). This is also calculated using Pythagoras' theorem to N_0 = 47.413 mm.

$$\alpha_i = \left. \frac{\partial M_0}{\partial M_i} \right|_{N_1,...,N_k} \quad \text{Eq. (6)}$$

$$\alpha_1 = \left. \frac{\partial M_0}{\partial M_1} \right|_{N_1,N_2,N_3,N_4}$$

$$\alpha_1 = \frac{\partial M_0}{\partial M_1} = -\frac{2M_2 + 2M_1}{2\left(\sqrt{M_2^2 - 2M_2M_1 + M_1^2 + M_4^2 - 2M_4M_3 + M_3^2}\right)} \quad \text{Eq. (7)}$$

$$\alpha_2 = \left. \frac{\partial M_0}{\partial M_2} \right|_{N_1,N_2,N_3,N_4}$$

$$\alpha_2 = \frac{\partial M_0}{\partial M_2} = \frac{2M_2 + 2M_1}{2\left(\sqrt{M_2^2 - 2M_2M_1 + M_1^2 + M_4^2 - 2M_4M_3 + M_3^2}\right)} \quad \text{Eq. (8)}$$

$$\alpha_3 = \left. \frac{\partial M_0}{\partial M_3} \right|_{N_1,N_2,N_3,N_4}$$

$$\alpha_3 = \frac{\partial M_0}{\partial M_3} = -\frac{2M_4 + 2M_3}{2\left(\sqrt{M_2^2 - 2M_2M_1 + M_1^2 + M_4^2 - 2M_4M_3 + M_3^2}\right)} \quad \text{Eq. (9)}$$

$$\alpha_4 = \left. \frac{\partial M_0}{\partial M_4} \right|_{N_1,N_2,N_3,N_4}$$

$$\alpha_4 = \frac{\partial M_0}{\partial M_4} = \frac{2M_4 + 2M_3}{2\left(\sqrt{M_2^2 - 2M_2M_1 + M_1^2 + M_4^2 - 2M_4M_3 + M_3^2}\right)} \quad \text{Eq. (10)}$$

$$N_0 = \sum_{i=1}^{k} \alpha_i \cdot N_i \quad \text{Eq. (11)}$$

Following clarification of the functional interdependencies, the tolerance analysis proper may be conducted. For this, initially the maximum arithmetical closure dimension P_O = 47.953 mm and the minimum closure dimension P_U = 46.873 mm are calculated by using equations (12) and (13). The difference is the arithmetical closure dimension tolerance T_a = 1.08 mm. The arithmetical closure dimension tolerance can also be calculated by using Eq. (14).

$$P_O = \sum_{i=1}^{n} |\alpha_i| \cdot G_{O_{pos_i}} - \sum_{j=1}^{m} |\alpha_j| \cdot G_{U_{neg_j}} \quad \text{Eq. (12)}$$

$$P_U = \sum_{i=1}^{n} |\alpha_i| \cdot G_{U_{pos_i}} - \sum_{j=1}^{m} |\alpha_j| \cdot G_{O_{neg_j}} \quad \text{Eq. (13)}$$

$$T_a = \sum_{i=1}^{k} |\alpha_i| \cdot t_{a_i} \quad \text{Eq. (14)}$$

The arithmetical result shows that the contact point of the two wheels is not constantly in the transition area of the two pitch circle radii, but may be distributed in a tolerance range of 1.08 mm. If applied to the functional dimension specification M_0 = 47.4 ± 0.3 mm, the arithmetical result with P_O = 47.953 mm and P_U = 46.873 mm is not correct. If dependable interlocking functionality is to be achieved the individual tolerances must be narrowed down further.

3 Statistical Tolerance Analysis of the Adjustment Mechanism

Given that – as early as the construction design phase – the eventual construction of the individual components and that the function of the sub-assemblies must be assured in and by the mass-produced components, a merely mathematical verification of the construction will not suffice. It is far more important to provide theoretical proof that the construction will fulfil the functional required of it. This can only be achieved by means of a statistical tolerance analysis.

In order to determine the statistical closure dimension tolerance T_s for the distance between shafts, production-specific effect parameters in the shape of production distribution (and thereby process parameters) are allocated to the four individual tolerances prior to calculation. It is assumed for the adjustment mechanism example that the probability density functions of the actual dimensions, generally known as production distributions, of the four closed dimension chains are distributed normally with an acceptance probability of P_a = 99.73 %, with the parameters as per **Table 1**.

Asymmetrical distribution patterns exist alongside the symmetrical distribution types shown in Table 1. Examples of asymmetrical patterns are logarithmic standard distributions for deviation from smooth running pertaining to surfaces with rotation symmetry, or Rayleigh spread in the case of eccentricity, concentricity or positional tolerances. Furthermore, hybrid spreads of the first or second kind can also develop [5].

Using Gauß' error reproduction with Eq. (15), standard deviation or spread margin of the distance between shafts M_0 may be calculated [4], [2]. This results, given the above-mentioned peripheral conditions, in σ_0 = 0.0942 mm. In equations (11) to (15) and in Eq. (17), it is important to ensure that the linearity coefficients are included, partly as a relative value and partly as an absolute value.

$$\sigma_0 = \sqrt{\sum_{i=1}^{k} \alpha_i^2 \cdot \sigma_i^2} \qquad \text{Eq. (15)}$$

$$T_s = 2 \cdot u \cdot \sigma_0 \qquad \text{Eq. (16)}$$

$$N_0 = \sum_{i=1}^{k} \alpha_i \cdot N_i \qquad \text{Eq. (17)}$$

Table 1: Production spread types and parameters [5]

Spread	Pa [%]	σ^2	cp	Quantile $u_{o/u}$
Rectangle	100	$\frac{t^2}{12}$	0.57735	± 1.732
Trapezium Side ratio 1/2 to 1	100	$\frac{10}{192} t^2$	0.73029	± 2.190
Trapezium Side ratio 1/3 to 1	100	$\frac{5}{108} t^2$	0.77459	± 2.323
Triangle	100	$\frac{t^2}{24}$	0.81649	± 2.449
Normal	99.73002	$\frac{t^2}{36}$	1.00000	± 3.0
Normal	99.9936	$\frac{t^2}{64}$	1.33333	± 4.0

Table 2: Evaluation of the torque key experiment series

Experiment series	d [mm]	Parameter l [mm]	M_t [Nmm]	Angle φ [°]	Gradient	Coefficient
1.1	12	800	100,000	28	positive	**α_l = 0.035**
1.2	12	900	100,000	31.5		
2.1	12	1,143	100,000	40	negative	**α_d = -11**
2.2	13	1,143	100,000	29		
2.1	12	1,143	100,000	40	positive	**α_{Mt} = 0.0005**
3.2	12	1,143	101,000	40.5		
2.1	12	1,143	100,000	40	negative	**α_d = -15**
4.2	12.1	1,143	100,000	38.5		

The method used here is based on the "central borderline axiom" of statistics. In this case, the sum of any given independent spreads numbering ≥ 4 is close enough to be considered a standard spread. According to the standardised spread definition, which has already been calculated for $\mu = 0$ and for $\sigma = 1$ [3], the quantile is u = ± 3.0 with an acceptance probability P_a = 99.73002 %, corresponding to a process capability c_p = 1.0. If u = ± 4.0 with an acceptance probability P_a = 99.9936 %, then this corresponds to the process capability c_p = 1.33. By using equations (15) and (16), the statistical closure dimension tolerance for process capability c_p = 1.0 with T_s = 0.565 mm may be calculated.

The resulting density function for the closure dimension tolerance is shown in **Figure 2**. It can be seen that that the statistical spread around the nominal closure dimension N_0 = 47.413 mm, calculated according to Eq. (17), is symmetrical at ± T_s/2.

The result of the statistical tolerance analysis shows that the function of the distance between shafts guarantees unproblematic interlocking despite the results of the mathematical calculation, as the tolerance area of 0.565 mm remains within the nominal value of ± 0.3 mm.

The example shown reflects many actual constructions which, following verification by a mathematical tolerance analysis, should not function properly but are put into practice nevertheless. The result is that serial production has complete or qualified process capability, as the laws of statistics apply here.

4 Empirical Determination of Linearity Coefficients

The example of the adjustment mechanism shows the resolution model of linearisation using the "total differential" to determine the linearity coefficients. The use of linearisation as set out in Eq. (5), however, assumes that the objective function for M_0 is known. The following empirical resolution model should be used whenever the functional interdependencies are not completely clear or cannot be completely described. The aim here is to use experimental constructions to discover the required information. The example of a torque key, as shown in **Figure 3**, has been chosen to describe the method here. In the case of the current example, the torque key must be dimensioned in such a way that the angle of twist is φ = 40° when the torsion moment is M_t = 100,000 Nmm.

The object is to calculate the resulting angle of twist φ of the torsion bar under the given manufacturing tolerance, firstly in order to calibrate the scale and secondly to check the accuracy of the torque key by reading from the scale the value for the force used.

The angle of twist will behave in a linear elastic range in proportion to the force used. The linearity coefficients are also required here in order to be able to calculate the angle of twist. As we already know, they assume a known mathematical objective function for the angle of twist.

Let it initially be assumed that this functional dependency is unknown. It follows that a statement from the point of view of the constructor must be made as to the values which have an effect. From the point of view of the constructor, the torsion moment M_t, the torsion bar diameter d and the torsion bar length l are values which are sure to have an effect.

If these are really the links of the closed dimension chain, the aim is to determine the required linearity coefficients for them. It is quite possible that values with an effect are, by their being unknown, neglected here, which should of course be avoided if possible. If values with an effect are, however, inadvertently neglected, this will not provide a false result for the dimensions in question.

Next, simple experiment apparatus is constructed in order to determine the linearity coefficients. The torsion bar is made from heat-treated steel 42CrMo4. Solid cylindrical material with a diameter d = 12 mm is selected. The torsion stress is calculated using equations (18) and (19) to τ_t = 294.73 N/mm², which is therefore less than the permitted stress of τ_t = 350 N/mm².

$$W_p = \frac{d^3 \cdot \pi}{16} \quad \text{Eq. (18)}$$

$$\tau_t = \frac{M_t}{W_p} \quad \text{Eq. (19)}$$

First the length of the torsion key bar is determined using the simple experiment apparatus. For this purpose, as shown in **Figure 4**, bars measuring 800, 900, 1,000, 1,100 und 1,200 mm made from the selected material in the corresponding diameter were manufactured. Then the various bars were subjected to a constant torsion moment of M_t = 100,000 Nmm in the apparatus and the resulting angles of twist measured. The assessment of the experiment in Figure 4 shows that the desired angle of twist φ = 40° is achieved between the bar lengths 1,100 and 1,200 mm. The assessment also shows that the angle of twist changes in proportion to the bar length, so that the bar length for φ = 40° with

l = 1,142.85 mm may be calculated by a simple rule of three according to Eq. (20). So now all nominal values for the torque key have been determined: bar diameter d = 12 mm, bar length l = 1,143 mm and torsion moment M_t = 100,000 Nmm.

$$l = \frac{1000\,mm \cdot 40°}{35°} = 1.142,85\,mm \quad \text{Eq. (20)}$$

The next experimental apparatus are to be constructed on the basis of the knowledge gained here with the purpose of determining the linearity coefficients of the three links of the closed dimension chain. As this resolution model describes an empirical linearisation, one link in the chain will be altered slightly in the course of the experiment, and the direct influence of this alteration on the function dimension determined or measured. This happens under the assumption that the nominal values of the other closed chain dimensions remain constant. The change in length of the closed dimension chain links should be as small as possible, as otherwise an error increase in the determination of the linearity coefficients occurs.

First the coefficient for the bar length is to be determined. The evaluated results from the first experimental apparatus are used for this, see Figure 4. The coefficient may be calculated using Eq. (21) at α_l = 0.035 °/mm, from the quotient of the change in angle divided by the bar length. At the same time, the function increase in Figure 4 shows whether it is a positive or a negative dimension chain link. In this case it is positive.

$$\alpha_l = \frac{31,5° - 28°}{100\,mm} = 0,035°/mm \quad \text{Eq. (21)}$$

A further experiment is conducted to determine the bar diameter coefficient by measuring the angle of twist subject to the nominal dimension structure. Following experiment series 2.1 this is φ = 40°. Experiment series 2.1 is used as a reference for the subsequent series, see **Table 2**.

Afterwards the torsion bar is replaced by one with a diameter d = 13 mm and the nominal torsion moment applied again. Now the angle of twist is only 29°, see **Figure 5**. The diameter coefficient is calculated accordingly using Eq. (22) at α_d = -11 °/mm. In this case, it is a negative dimension chain link.

$$\alpha_d = \frac{29° - 40°}{1\,mm} = -11°/mm \quad \text{Eq. (22)}$$

Finally the coefficient for the torsion moment is determined. On the basis of the result of experiment series 2.1, the torsion moment is increased from 100,000 to 101,000 Nmm with the result that an angle of twist of φ = 40.5° is achieved, see **Figure 6**. Following Eq. (23), the resulting coefficient for the torsion moment is α_{Mt} = 0.0005 °/Nmm. This chain link is positive again.

$$\alpha_{M_t} = \frac{40,5° - 40°}{1000\,Nmm} = 0,0005°/\text{Nmm} \quad \text{Eq. (23)}$$

The results of the experiment series show that the bar diameter has a significant effect on the resulting angle of twist. As the empirical determination of the coefficients involves a linearisation, as we have already explained, the difference should be as small as possible, especially in the case of significant links. Therefore a further experiment with a bar diameter d = 12.1 mm is conducted, see **Figure 7**. The result of Eq. (24) with α_d = -15 °/mm justifies the new experiment, as the coefficient has again changed significantly.

$$\alpha_d = \frac{38,5° - 40°}{0,1\,mm} = -15°/mm \quad \text{Eq. (24)}$$

Now that the function dependency has been clarified by way of the linearity coefficients, the tolerance analysis proper may be carried out. Thus he arithmetic closure dimension tolerance for the angle of twist is calculated with the tolerances for the three dimension chain links given in Figure 3, following Eq. (14), with T_a = 7.084°.

For statistical tolerance analysis purposes, the manufacturing qualities as in Table 1 are allocated as follows: the actual measurement spread over the area of tolerance of the torsion moment should correspond to a rectangular spread pattern, the actual measurement spread for the bar length to a trapezium with a side ratio of 0.5/1 t and the actual measurement tolerance spread for the bar diameter should correspond to a normal spread pattern with an acceptance probability of P_a = 99.73 %.

Using equations (15) and (16), the statistical closure dimension tolerance for P_a = 99.73 % with T_s = 6.246° may be calculated. According to this, the deviation at an angle of twist of φ = 40° will only come to 88.1 % of the arithmetical result. If the diameter tolerance were only ± 0.05 instead of ± 0.2 mm, then the statistically determined deviation from the nominal angle would only be ± 1.145° instead of ± 3.12°. Use of this method for empirical linearisation shows how easy and relatively quick processing can be, especially where physical systems are complex and/or arbitrary.

5 Calculation of the Torque Key Using Linearisation

In order to prove the the coefficients obtained empirically, linearisation should also be conducted using “total differential“. For the empirical determination of the coefficients it was assumed that the objective function fort he angle of twist was unknown. In fact the angle of twist can be calculated using Eq. (25), however [7]. The polar moment of inertia I_p according to Eq. (26) must be considered here. Provided that the torsion bar cross-section consists of a completely solid, cylindrical material, the angle of twist $M_0 = f(M_t, d, l, G)$ depends on the applied torsion moment M_t, the torsion bar diameter d, the torsion bar length l and the slip module G of the torsion bar.

$$M_0 = \varphi = \frac{M_t \cdot l}{G \cdot I_p} \cdot \frac{180°}{\pi} \quad \text{Eq. (25)}$$

$$I_p = \frac{d^4 \cdot \pi}{32} \quad \text{Eq. (26)}$$

If the objective function for M_0 is known, the solution method of linearisation following Eq. (5) can be used to determine linearity coefficients. The 1st order partial derivations then give the following linearity coefficients for the torsion moment following Eq. (27) $\alpha_{Mt} = 4.00007589\ 10^{-4}$, the bar length following Eq. (28) $\alpha_l = 0.035180966$, the bar diameter following Eq. (29) α_d = -13.33358631, and the slip module according to Eq. (30) $\alpha_G = -5.00009486\ 10^{-4}$.

$$\alpha_{M_t} = \frac{\partial M_O}{\partial M_t}\Big|_{N_{M_t}, N_l, N_G, N_d}$$

$$\alpha_{M_t} = \frac{\partial}{\partial M_t}\left(\frac{M_t \cdot 1 \cdot 180° \cdot 32}{G \cdot d^4 \cdot \pi^2}\right) \Big|_{N_{M_t}, N_l, N_G, N_d} = 4{,}00007589 \cdot 10^{-4} \quad \text{Eq. (27)}$$

$$\alpha_l = \frac{\partial M_0}{\partial l}\Big|_{N_{M_t}, N_l, N_G, N_d} = 0{,}035180966 \quad \text{Eq. (28)}$$

$$\alpha_d = \frac{\partial M_0}{\partial d}\Big|_{N_{M_t}, N_l, N_G, N_d} = -13{,}33358631 \quad \text{Eq. (29)}$$

$$\alpha_G = \frac{\partial M_0}{\partial G}\Big|_{N_{M_t}, N_l, N_G, N_d} = -5{,}00009486 \cdot 10^{-4} \quad \text{Eq. (30)}$$

The linearity coefficients obtained by empirical methods correspond surprisingly well with those calculated here using linearisation, see Table 2. As Eq. (27) shows, the slip module was unknowingly not taken into consideration in the empirical resolution model. The other coefficients obtained are nevertheless correct, despite this omission.

This is an excellent demonstration that all or at least a significant number of construction design parameters can be discovered by experimental means, without knowledge of the functional interdependencies. This is of decisive relevance in the case of arbitrary physical closed dimension chains, as shown by the torque key example.

6 Summary

The process for empirical determination of the linearity coefficients showed that, with the simplest of apparatus and without prior knowledge of the objective function, it may be used for sub-assembly function. Even when important or less important parameters are overlooked in the course of verification the recorded coefficients are still correct. The arithmetical tolerance analysis may be conducted subsequently, in the knowledge of the coefficients determined in this way. Using these results as a basis, the statistical tolerance analysis may be conducted with the help of the central limit theorem or error propagation according to Gauß. The aim here is to find reliable and validated functional dimensions as early as possible during the design phase.

Many companies from the field of machine and vehicle manufacturing or of electronics have already identified the increasing usefulness of tolerance analysis as an important factor in remaining at the top of the competitiveness pyramid. Here, using the laws of statistics, essential manufacturing qualities are rendered less tolerance-sensitive and, at the same time, more process-oriented tolerances for a functional dimension are defined.

The definition of process-oriented tolerances also entails process-oriented thoughts and actions. The positive results of this are that problems are brought to light and therefore decisions about how to solve them made earlier, which saves both time and money.

Symbols

α_i	linearity coefficient	N_0	nominal value of closure dimension
φ	angle of twist	P_a	acceptance probability
τ_t	torsion stress	P_O	maximum closure dimension (highest possible value for good fit)
σ^2	variance	P_U	minimum closure dimension (lowest possible value for good fit)
σ_0	standard deviation of the functional dimension	t_{ai}	arithmetical closed dimension chain link tolerance
d	torsion bar diameter	T_a	arithmetical closure dimension tolerance
G_O	maximum dimension	T_s	statistical closure dimension tolerance
G_U	minimum dimension	u	acceptance probability in s-units of standardised normal spread
G	slip module	W_p	polar resistance moment
I_p	polar moment of inertia		
l	torsion bar length		
M_i	tolerated dimension		
M_0	closure- or functional dimension		
M_t	torsion moment		
k, n, m	number of dimension chain links		

References

[1] DIN 7186 Blatt 1, Statistische Tolerierung – Begriffe, Anwendungsrichtlinien und Zeichnungsangaben, Berlin: Beuth 1974

[2] DIN 7186 Blatt 2 (Entwurf), Statistische Tolerierung – Grundlagen für Rechenverfahren, Berlin: Beuth 1980

[3] Mannewitz, F.: Prozessfähige Tolerierung von Bauteilen und Baugruppen. Ein Lösungsansatz zur Optimierung der Werkstattfertigung im Informationsverbund zwischen CAD und CAQ. In: VDI-Fortschrittsberichte, Reihe 20, Nr. 256, 1997

[4] Klein, B.; Mannewitz, F.: Statistische Tolerierung. Braunschweig/Wiesbaden: Vieweg-Verlag, 1993

[5] Simunovic, M; Mannewitz, F.: Schulungsunterlagen: Statistische Tolerierung in der Serienfertigung. Casim Ingenieurleistungen, 2002

[6] Papula, L.: Mathematische Formelsammlung für Ingenieure und Naturwissenschaftler. Braunschweig/Wiesbaden: Vieweg-Verlag, 1992

[7] Böge, A.: Mechanik und Festigkeitslehre. Braunschweig/Wiesbaden: Friedr. Vieweg & Sohn-Verlag, 1984, S. 274-277