Tolerance Analyses of Arbitrary Physical Systems

Tolerance analyses are indispensable to prove the technical practicability of sub-assemblies under mass production conditions. This contribution is intended as an attempt to show an empirical resolution method which allows the easy implementation of complex tolerance analyses on arbitrary physical systems.

1 Introduction

Tolerance investigations, frequently also termed tolerance analyses, have been indispensable ever since the beginning of the last century, starting with Henry Ford’s pioneering work. Their purpose is to prove the technical practicability of sub-assemblies under mass production conditions. However, in recent years the subject of tolerance analyses on technical sub-assemblies has gained more and more significance, not least due to the fact that tolerance analyses can determine tolerance margins and, by implication, manufacturing costs.

The calculation of manufacturing tolerances has become a central theme for the motor vehicle industry and its sub-contractors alike, as the competitiveness of a company is decided by quality requirements as well as time requirements.

For a tolerance analysis to be carried out, the functional interdependencies of a sub-assembly or technical system must be known. That is to say that it must also be known which links form the closed chain and, above all, what influence each individual link of the chain has on the function of the sub-assembly.

Besides describing the general methods used in tolerance analysis, this contribution is intended to demonstrate an empirical resolution method for determining linearity coefficients. This method consists of simple experiments to show functional interdependencies quickly and easily and thus facilitate the implementation of tolerance analyses. This paper will, therefore, include statistical analysis as well as arithmetical analysis.

2 Description of Closed Dimension Chains

Figure 1 shows part of the forward section of a metal housing used for adjusting an automobile air conditioning system. The illustration depicts the bearing of the drive- and transmission cog for the ventilation flap. In
general, transmission of force in a gear system (in this case an involute gear) can only be guaranteed if the meshing of the two wheels is harmonised in such a way that the sum of the two pitch circle radii is equal to the dimension required \( M_0 \) between the centre of the shafts.

Therefore the dimension \( M_0 \), also called closure dimension, necessary to ensure functionality is fixed at 47.4 ± 0.3 mm. If the closure dimension is too small, the teeth heads can protrude into the teeth bases of the opposite cog. If the closure dimension is too large, the cogs may not run smoothly or even fail to interlock entirely.

The tolerances for form and position normally required for a correct function description are not taken into account in the calculation example. This should be taken into account, based on the appropriate tolerance principle and tolerance type.

The individual dimensions which affect the distance between shafts \( M_0 \) in this case are the horizontal and vertical positions of the bearing centres in the metal housing structure relative to their reference faces.

Since the technical implementation of the example shown here should be realised with a double perforation operation, it is necessary to consider the respective X- and Y-coordinates separately. These are a result of the tool precision on the one hand, and of the exact insertion of the metal into the tool on the other. The holes for the drive wheel bearing are punched during the first stage and those for the transmission wheel during the second. Consequently the effective tolerated individual dimensions of the distance between shafts \( M_0 \) are the horizontal lengths \( M_1 \) and \( M_2 \) and the vertical lengths \( M_3 \) and \( M_4 \).

So the nominal dimension of the distance between the two shafts under the nominal dimensions given in Figure 1 is calculated according to Pythagoras’ theorem following Eq. (1) to \( N_0 = 47.413 \text{ mm} \). Eq. (1) describes (allowing for the respective tolerated individual dimension \( M_i \)) the objective function of the sought dimension \( M_0 \), as Eq. (2) shows. Generally speaking a closed dimension chain is a function of the respective tolerated individual dimensions \( M_i \) on the closure dimension \( M_0 \) are recorded via so-called linearity coefficients \( \alpha_i \). If the linearity coefficients of a dimension chain are exclusively ± 1, it is known as a one-dimensional or linear dimension chain [1]. So a dimension chain in general can be described by Eq. (3).

\[
M_0 = \sum_{i=1}^{n} \alpha_i \cdot M_i \tag{3}
\]

The general resolution method to determine linear coefficients is given in the linearisation of functions using the “total differential”[6]. In this case the function \( y = f(x_1, x_2, ..., x_k) \) in the immediate vicinity of the centre of area \( (x_1, x_2, ..., x_k) \) is replaced by a linear function, that is to say, the total differential of the function, see Eq. (4).

\[
\Delta y = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \cdot \Delta x_i \tag{4}
\]

\[
dM_0 = \sum_{i=1}^{k} \left( \alpha_i \cdot \frac{\partial M_i}{\partial x_i} \right) \tag{5}
\]

The first order partial deductions are formed for the centre of area \( P \). Also the \( \Delta x_i \) are the “minor” deviations with reference to the centre of area. As a result, Eq. (5) follows by using Eq. (6). The respective nominal dimensions are considered in the derivations. Using Eq. (6), the following deductions result for the adjustment mechanism example according to equations (7) to (10), and therefore linearity coefficients \( \alpha_1 = -0.46400699, \alpha_2 = 0.46400699, \alpha_3 = -0.88583153 \) and \( \alpha_4 = 0.88583153 \). When the linearity coefficients are known, the nominal closure dimension \( N_0 \) may be calculated according to Eq. (11). This is also calculated using Pythagoras’ theorem to \( N_0 = 47.413 \text{ mm} \).

\[
\alpha_i = \frac{\partial M_i}{\partial M_0} \tag{6}
\]

\[
\alpha_1 = \frac{\partial M_1}{\partial M_0} = \frac{2M_1 + 2M_2}{\left(\sqrt{M_1^2 + 2M_1 M_2 + M_2^2} \right)} \tag{7}
\]

\[
\alpha_2 = \frac{\partial M_2}{\partial M_0} = \frac{2M_2 + 2M_1}{\left(\sqrt{M_1^2 + 2M_1 M_2 + M_2^2} \right)} \tag{8}
\]

\[
\alpha_3 = \frac{\partial M_3}{\partial M_0} = \frac{2M_3 + 2M_1}{\left(\sqrt{M_1^2 + 2M_1 M_2 + M_2^2} \right)} \tag{9}
\]

\[
\alpha_4 = \frac{\partial M_4}{\partial M_0} = \frac{2M_4 + 2M_1}{\left(\sqrt{M_1^2 + 2M_1 M_2 + M_2^2} \right)} \tag{10}
\]

\[
N_0 = \sum_{i=1}^{n} \alpha_i \cdot N_i \tag{11}
\]

Following clarification of the functional interdependencies, the tolerance analysis proper may be conducted. For this, initially the maximum arithmetical closure dimension \( P_{0} = 47.953 \text{ mm} \) and the minimum closure dimension \( P_{0} = 46.873 \text{ mm} \) are calculated by using equations (12) and (13). The difference is the arithmetical closure dimension tolerance \( T_a = 1.08 \text{ mm} \). The arithmetical closure dimension tolerance can also be calculated by using Eq. (14).

\[
P_0 = \sum_{i=1}^{n} \alpha_i \cdot |G_{ij}| \cdot T_a \tag{12}
\]

\[
P_0 = \sum_{i=1}^{n} \alpha_i \cdot |G_{ij}| \cdot T_a \tag{13}
\]

\[
T_a = \sum_{i=1}^{n} \alpha_i \cdot T_a \tag{14}
\]

The arithmetical result shows that the contact point of the two wheels is not constantly in the transition area of the two pitch circle radii, but may be distributed in a tolerance range of 1.08 mm. If applied to the functional dimension specification \( M_0 = 47.4 \pm 0.3 \text{ mm} \), the arithmetical result with \( P_0 = 47.953 \text{ mm} \) and \( P_0 = 46.873 \text{ mm} \) is not correct. If dependable interlocking function-
3 Statistical Tolerance Analysis of the Adjustment Mechanism

Given that – as early as the construction design phase – the eventual construction of the individual components and that the function of the sub-assemblies must be assured in and by the mass-produced components, a merely mathematical verification of the construction will not suffice. It is far more important to provide theoretical proof that the construction will fulfill the functional required of it. This can only be achieved by means of a statistical tolerance analysis.

In order to determine the statistical closure dimension tolerance $T_c$ for the distance between shafts, production-specific effect parameters in the shape of production distribution (and thereby process parameters) are allocated to the four individual tolerances prior to calculation. It is assumed for the adjustment mechanism example that the probability density functions of the actual dimensions, generally known as production distributions, of the four closed dimension chains are distributed normally with an acceptance probability $P_a = 99.73 \%$, with the parameters as per Table 1.

Asymmetrical distribution patterns exist alongside the symmetrical distribution types shown in Table 1. Examples of asymmetrical patterns are logarithmic standard distributions for deviation from smooth running pertaining to surfaces with rotation symmetry, or Rayleigh spread in the case of eccentricity, concentricity or positional tolerances. Furthermore, hybrid spreads of the first or second kind can also apply here.

Using Gauß’ error reproduction with Eq. (15), standard deviation or spread margin of the distance between shafts $M_0$ may be calculated [4], [2]. This results, given the above-mentioned peripheral conditions, in $\sigma_\alpha = 0.0942 \text{ mm}$. In equations (11) to (15) and in Eq. (16), the statistical closure dimension $N_0 = 47.413 \text{ mm}$, calculated according to Eq. (17), is important to ensure that the linearity coefficients are included, partly as a relative value and partly as an absolute value.

$$\sigma_\alpha = \sqrt{\sum_{i=1}^{4} \alpha_i^2 \cdot \sigma_i^2} \quad \text{Eq. (15)}$$

$$T_c = 2 \cdot u \cdot \sigma_\alpha \quad \text{Eq. (16)}$$

$$N_0 = \sum_{i=1}^{4} \alpha_i \cdot N_i \quad \text{Eq. (17)}$$

Table 1: Production spread types and parameters [5]

<table>
<thead>
<tr>
<th>Spread</th>
<th>$P_a$ (%)</th>
<th>$\sigma_\alpha$</th>
<th>$c_p$</th>
<th>Quantile $u_{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>100</td>
<td>$\frac{t}{12}$</td>
<td>0.57735</td>
<td>$\pm 1.732$</td>
</tr>
<tr>
<td>Trapezium side ratio 1/2 to 1</td>
<td>100</td>
<td>$\frac{10}{192}$</td>
<td>0.73029</td>
<td>$\pm 2.190$</td>
</tr>
<tr>
<td>Trapezium side ratio 1/3 to 1</td>
<td>100</td>
<td>$\frac{5}{108}$</td>
<td>0.77459</td>
<td>$\pm 2.323$</td>
</tr>
<tr>
<td>Triangle</td>
<td>100</td>
<td>$\frac{t}{23}$</td>
<td>0.81649</td>
<td>$\pm 2.449$</td>
</tr>
<tr>
<td>Normal</td>
<td>99.73002</td>
<td>$\frac{t}{36}$</td>
<td>1.00000</td>
<td>$\pm 3.0$</td>
</tr>
<tr>
<td>Normal</td>
<td>99.9836</td>
<td>$\frac{t}{64}$</td>
<td>1.33333</td>
<td>$\pm 4.0$</td>
</tr>
</tbody>
</table>

The example shown reflects many actual constructions which, following verification by a mathematical tolerance analysis, should not function properly but are put into practice nevertheless. The result is that serial production has complete or qualified process capability, as the laws of statistics apply here.

4 Empirical Determination of Linearity Coefficients

The example of the adjustment mechanism shows the resolution model of linearisation using the “total differential” to determine the linearity coefficients. The use of linearisation as set out in Eq. (5), however, assumes that the objective function for $M_0$ is known. The following empirical resolution model should be used whenever the functional interdependencies are not completely clear or cannot be completely described. The aim here is to use experimental constructions to discover the required information. The example of a torque key, as shown in Figure 3, has been chosen to describe the method here. In the case of the current example, the torque key must be dimensioned in such a way that the angle of twist $\varphi$ is $40^\circ$ when the torsion moment is $M_t = 100,000 \text{ Nmm}$.

The object is to calculate the resulting angle of twist $\varphi$ of the torsion bar under the given manufacturing tolerance, firstly in or-
order to calibrate the scale and secondly to check the accuracy of the torque key by reading from the scale the value for the force used.

The angle of twist will behave in a linear elastic range in proportion to the force used. The linearity coefficients are also required here in order to be able to calculate the angle of twist. As we already know, they assume a known mathematical objective function for the angle of twist.

Let it initially be assumed that this functional dependency is unknown. It follows that a statement from the point of view of the constructor must be made as to the values which have an effect. From the point of view of the constructor, the torsion moment \( M_t \), the torsion bar diameter \( d \) and the torsion bar length \( l \) are values which are sure to have an effect.

If these are really the links of the closed dimension chain, the aim is to determine the required linearity coefficients for them. It is quite possible that values with an effect are, by their being unknown, neglected here, which should of course be avoided if possible. If values with an effect are, however, inadvertently neglected, this will not provide a false result for the dimensions in question.

Next, simple experiment apparatus is constructed in order to determine the linearity coefficients. The torsion bar is made from heat-treated steel 42CrMo4. Solid cylindrical material with a diameter \( d = 12 \text{ mm} \) is selected. The torsion stress is calculated using Eqs. (21) and (22) from the quotient of the change in angle divided by the bar length. At the same time, the function increase in Figure 4 shows whether it is a positive or a negative dimension chain link. In this case it is positive.

First the coefficient for the bar length is to be determined. The evaluated results from the first experimental apparatus are used for this, see Figure 4. The coefficient may be calculated using Eq. (21) at \( \alpha_l = 0.035 \degree/\text{mm} \), from the quotient of the change in angle divided by the bar length. At the same time, the function increase in Figure 4 shows whether it is a positive or a negative dimension chain link. In this case it is positive.

A further experiment is conducted to determine the bar diameter coefficient by measuring the angle of twist subject to the nominal dimension structure. Following experiment series 2.1 this is \( \varphi = 40\degree \). Experiment series 2.1 is used as a reference for the subsequent series, see Table 2.

Afterwards the torsion bar is replaced by one with a diameter \( d = 13 \text{ mm} \) and the nominal torsion moment applied again. Now the angle of twist is only \( 29\degree \), see Figure 5. The diameter coefficient is calculated according to Eq. (22) at \( \alpha_d = -0.005\degree/\text{mm} \). In this case, it is a negative dimension chain link.

Finally the coefficient for the torsion moment is determined. On the basis of the result of experiment series 2.1, the torsion moment is increased from 100,000 to 101,000 Nmm with the result that an angle of twist of \( \varphi = 40.5\degree \) is achieved, see Figure 6. Following Eq. (23), the resulting coefficient for the torsion moment is \( \alpha_{M_t} = 0.0005\degree/\text{Nmm} \). This chain link is positive again.

The results of the experiment series show that the bar diameter has a significant effect on the resulting angle of twist. As the empirical determination of the coefficients involves a linearisation, as we have already explained, the difference should be as small as possible, especially in the case of significant links. Therefore a further experiment with a bar diameter \( d = 12.1 \text{ mm} \) is conducted, see Figure 7. The result of Eq. (24) with \( \alpha_l = -15\degree/\text{mm} \) justifies the new experiment, as the coefficient has again changed significantly.

Now that the function dependency has been clarified by way of the linearity coefficients, the tolerance analysis proper may be carried out. Thus the arithmetic closure dimension tolerance for the angle of twist is calculated with the tolerances for the three dimension chain links given in Figure 3, following Eq. (14), with \( T_\varphi = 7.084\degree \).

For statistical tolerance analysis purposes, the manufacturing qualities as in Table 1 are allocated as follows: the actual measurement spread over the area of tolerance of the torsion moment should correspond to a rectangular spread pattern, the actual measurement spread for the bar length to a trapezium with a side ratio of 0.5/1 and the actual measurement tolerance spread for the bar diameter should correspond to a normal spread pattern with an acceptance probability of \( P_\alpha = 99.73\% \).

Using equations (15) and (16), the statistical closure dimension tolerance for \( P_\alpha = 99.73\% \) with \( T_{\alpha} = 6.246\degree \) may be calculated. According to this, the deviation at an angle of twist of \( \varphi = 40\degree \) will only come to 88.1\% of the arithmetical result. If the diameter tolerance were only \( \pm 0.05 \) instead of \( \pm 0.2 \) mm, then the statistically determined deviation from the nominal angle would only be \( \pm 1.145\degree \) instead of \( \pm 3.12\degree \). Use of this method for empirical linearisation shows how easy and relatively quick processing can be, especially where physical systems are complex and/or arbitrary.
5 Calculation of the Torque Key Using Linearisation

In order to prove the the coefficients obtained empirically, linearisation should also be conducted using “total differential”. For the empirical determination of the coefficients it was assumed that the objective function for the angle of twist was unknown. In fact the angle of twist can be calculated using Eq. (25), however [7]. The polar moment of inertia \( I_p \) according to Eq. (26) must be considered here. Provided that the torsion bar cross-section consists of a completely solid, cylindrical material, the angle of twist \( \frac{\partial M_0}{\partial t} = \frac{f(M_t, d, l, G)}{G \cdot \frac{d^4}{32}} \) depends on the applied torsion moment \( M_t \), the torsion bar diameter \( d \), the torsion bar length \( l \) and the slip module \( G \) of the torsion bar.

\[
M_0 = \varphi = \frac{M_t \cdot l}{G \cdot I_p} \quad 180^\circ \quad \text{Eq. (25)}
\]

\[
I_p = \frac{d^4 \cdot \pi}{32} \quad \text{Eq. (26)}
\]

If the objective function for \( M_0 \) is known, the solution method of linearisation following Eq. (5) can be used to determine linearity coefficients. The 1st order partial derivations then give the following linearity coefficients for the torsion moment following Eq. (27) \( \alpha_{M_0} = 4.00007589 \times 10^4 \), the bar length following Eq. (28) \( \alpha_t = 0.035180966 \), the bar diameter following Eq. (29) \( \alpha_d = -13.33358631 \), and the slip module according to Eq. (30) \( \alpha_G = -5.00009486 \times 10^4 \).

\[
\alpha_{M_0} = \frac{\partial M_0}{\partial M_t} = \frac{M_t \cdot l}{G \cdot I_p} \quad \text{Eq. (27)}
\]

\[
\alpha_t = \frac{\partial M_0}{\partial l} = \frac{M_t \cdot d^4}{G \cdot \frac{d^4}{32}} \quad \text{Eq. (28)}
\]

\[
\alpha_d = \frac{\partial M_0}{\partial d} = -\frac{M_t \cdot l}{G \cdot \frac{d^4}{32}} \quad \text{Eq. (29)}
\]

\[
\alpha_G = \frac{\partial M_0}{\partial G} = -\frac{M_t \cdot l}{G^2} \quad \text{Eq. (30)}
\]

The linearity coefficients obtained by empirical methods correspond surprisingly well with those calculated here using linearisation, see Table 2. As Eq. (27) shows, the slip module was unknowingly not taken into consideration in the empirical resolution model. The other coefficients obtained are nevertheless correct, despite this omission.

This is an excellent demonstration that all or at least a significant number of construction design parameters can be considered experimentally, without knowledge of the functional interdependencies. This is of decisive relevance in the case of arbitrary physical closed dimension chains, as shown by the torque key example.

6 Summary

The process for empirical determination of the linearity coefficients showed that, with the simplest of apparatus and without prior knowledge of the objective function, it may be used for sub-assembly function. Even when important or less important parameters are overlooked in the course of verification the recorded coefficients are still correct. The arithmetical tolerance analysis may be conducted subsequently, in the knowledge of the coefficients determined in this way. Using these results as a basis, the statistical tolerance analysis may be conducted with the help of the central limit theorem or error propagation according to Gauß. The aim here is to find reliable and validated functional dimensions as early as possible during the design phase.

Many companies from the field of machine and vehicle manufacturing or of electronics have already identified the increasing usefulness of tolerance analysis as an important factor in remaining at the top of the competitiveness pyramid. Here, using the laws of statistics, essential manufacturing qualities are rendered less tolerance-sensitive and, at the same time, more process-oriented tolerances for a functional dimension are defined.

The definition of process-oriented tolerances also entails process-oriented thoughts and actions. The positive results of this are that problems are brought to light and therefore decisions about how to solve them made earlier, which saves both time and money.

References